① Definition of sup / inf : S is <u>nonempty</u> subset of $\mathbb{R}$

② Q ~~3~~ 4 $\sup_y \inf_x h(x,y) \le \inf_x \sup_y h(x,y)$ ← e.g. ↑ check

since $h(x,y)$ is bounded
⇒ Thm 2.3.6
every nonempty set of real numbers that has upper bounds also has a supremum in $\mathbb{R}$
⇒ $\sup_y h(x,y)$ non empty

strict inequality , one example $h(x,y) := \begin{cases} 0 & \text{if } x < y \\ 1 & \text{if } x \ge y \end{cases}$

$\inf_x h(x,y) = 0$, $\sup_y \inf_x h(x,y) = 0$

$\neq$

$\sup_y h(x,y) = 1$, $\inf_x \sup_y h(x,y) = 1$

equality hold , one example $h(x,y) = 2x + y$, $0 \le x, y < 1$

$\inf_x h(x,y) = y$, $\sup_y \inf_x h(x,y) = 1$

$\parallel$

$\sup_y h(x,y) = 2x+1$, $\inf_x \sup_y h(x,y) = 1$

③ Definition (a) A set S is said to be countably infinite (denumerable) if there exists a bijection of $\mathbb{N}$ onto S

(b) A set S is said to be countable if it is either finite or countably infinite.

(c) A set S is said to be uncountable if it is not countable.

④ Theorem 2.5.5 The unit interval $[0,1] := \{x \in \mathbb{R} : 0 \le x \le 1\}$ is not countable.

Pf: Use contradiction

Assume $[0,1]$ is countable

Then $\exists\ x_n,\ n \in \mathbb{N}$ s.t. $\{x_1, x_2, \dots\} = [0,1]$

Also, fact: $\forall$ real number $x \in [0,1]$,

x has a decimal representation $x = 0.b_1 b_2 b_3 \dots$,

where $b_i = 0, 1, \dots, 9$

Then we have
$$x_1 = 0.b_{11} b_{12} \dots b_{1n} \dots,$$
$$x_2 = 0.b_{21} b_{22} \dots b_{2n} \dots,$$
$$\vdots$$
$$x_n = 0.b_{n1} b_{n2} \dots b_{nn} \dots,$$
$$\vdots \quad \dots$$

Now, we define real number $y := 0.y_1 y_2 \dots y_n$ by

setting $y_1 := \begin{cases} 2 & \text{if } b_{11} \geq 5 \\ 7 & \text{if } b_{11} \leq 4 \end{cases}$

in general,
$$y_n := \begin{cases} 2 & \text{if } b_{nn} \geq 5 \\ 7 & \text{if } b_{nn} \leq 4 \end{cases}$$

Then $y \in [0,1]$

First, $y \neq 0$ or 1, $\because y_n \neq 0, 9\ \forall n$

Second, For any $x_n,\ n \in \mathbb{N}$

$\because y_n \neq b_{nn}$

$\therefore y \neq x_n\ \forall n \in \mathbb{N}$

$\therefore y \notin \{x_1, x_2, \dots, x_n, \dots\} = [0,1]$

contradiction to $y \in [0,1]$

1. Find the limit of following sequences by definition

(a) $\lim_n \left(\frac{n}{n^2+1} \right)$

(b) $\lim_n \frac{2n}{n+2}$

2. Show that $\lim (\sqrt{n^2+1} - n) = 0$

3. Show that if $x_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim (x_n) = 0$ then $\lim (\sqrt{x_n}) = 0$

Definition: A sequence $X = (x_n)$ in $\mathbb{R}$ is said to converge to $x \in \mathbb{R}$ or x is said to be a limit of (x_n), if for every $\varepsilon > 0$, there exists a natural number $K(\varepsilon)$ such that for all $n \geq K(\varepsilon)$, the terms x_n satisfy $|x_n - x| < \varepsilon$.
If a sequence has a limit, we say that the sequence is convergent; If it has no limit, we say that the sequence is divergent.

Pf 1 a) Let $\varepsilon > 0$.

Look that $\frac{n}{n^2+1} < \frac{n}{n^2} \leq \frac{1}{n}$

Now choose $K \in \mathbb{N}$ s.t. $\frac{1}{K} < \varepsilon$. by Archimedean Property 2.4.3

Then $\forall n \geq k \Rightarrow \frac{1}{n} < \varepsilon$,

and $\left| \frac{n}{n^2+1} - 0 \right| = \frac{n}{n^2+1} < \frac{1}{n} < \varepsilon$

$\therefore$ 0 is limit of $\left\{ \frac{n}{n^2+1} \right\}$

1 b) Want to show $\lim_n \frac{2n}{n+2} = 2$

Let $\varepsilon > 0$, we want to show $\left|\frac{2n}{n+2} - 2\right| < \varepsilon$

$$\left|\frac{2n}{n+2} - 2\right| = \left|\frac{2n - 2n - 4}{n+2}\right| = \left|\frac{-4}{n+2}\right| = \frac{4}{n+2} < \frac{4}{n}.$$

Choose $K \in \mathbb{N}$ s.t. $\frac{1}{K} < \frac{\varepsilon}{4}$

$\forall\ n \geq K$, we have $\frac{1}{n} < \frac{\varepsilon}{4}$

Then $\left|\frac{2n}{n+2} - 2\right| < \frac{4}{n} < 4 \cdot \frac{\varepsilon}{4} = \varepsilon$

$\therefore \lim_n \frac{2n}{n+2} = 2$

2. Want to show $\lim_n \left(\sqrt{n^2+1} - n\right) = 0$

Let $\varepsilon > 0$. We want to show $\left|\sqrt{n^2+1} - n - 0\right| < \varepsilon$

$$\left|\sqrt{n^2+1} - n - 0\right| = \left|\frac{(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)}{\sqrt{n^2+1} + n}\right|$$

$$= \left|\frac{n^2+1-n^2}{\sqrt{n^2+1}+n}\right|$$

$$= \frac{1}{\sqrt{n^2+1}+n}$$

$$< \frac{1}{2n}$$

Choose $K \in \mathbb{N}$ s.t. $\frac{1}{K} < 2\varepsilon$

$\forall\ n \geq K$, we have $\frac{1}{n} < 2\varepsilon$

Then $\left|\sqrt{n^2+1} - n - 0\right| < \frac{1}{2n} < \frac{1}{2} \cdot 2\varepsilon = \varepsilon$

$\therefore \lim_n \left(\sqrt{n^2+1} - n\right) = 0$

Let $\varepsilon > 0$. We want to show $|\sqrt{x_n} - 0| < \varepsilon$

Pf 3. $\lim_n x_n = 0$

$\exists K \in \mathbb{N}$ s.t. if $n \geq K$, we have $|x_n - 0| < \varepsilon$

$$|\sqrt{x_n} - 0| = |\sqrt{x_n}|$$

$$= \sqrt{|x_n|} \quad \because x_n \geq 0$$

Think $\sqrt{|x_n|} < \varepsilon \Leftrightarrow |x_n| < \varepsilon^2$

$\lim_n x_n = 0$

$\Rightarrow \exists K \in \mathbb{N}$ s.t. if $n \geq K$, we have $|x_n - 0| < \varepsilon^2$

$\Rightarrow |x_n| < \varepsilon^2$

Thus we choose K as above

if $n \geq K$, $|\sqrt{x_n} - 0| = \sqrt{|x_n|} < \sqrt{\varepsilon^2} = \varepsilon$

Then $\lim_n |\sqrt{x_n} - 0| < \varepsilon$